Tropical Analysis
With an Application to Indivisible Goods

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Abstract

We establish the *Subgradient Theorem* for monotone correspondences – a monotone correspondence is equal to the subdifferential of a potential if and only if it is conservative, i.e. its integral along a closed path vanishes irrespective of the selection from the correspondence along the path. We prove two attendant results: the *Potential Theorem*, whereby a conservative monotone correspondence can be integrated up to a potential, and the *Duality Theorem*, whereby the potential has a dual whose subdifferential is a conservative monotone correspondence that is the inverse of the original correspondence. We use these results to reinterpret and extend Baldwin and Klemperer's (2019) characterization of demand in economies with indivisible goods.

Keywords: *Conservative correspondences, subgradient theorem, potential theorem, duality, tropical geometry, convex analysis, normally labeled polyhedral subdivisions, subdifferentials, indivisible goods*

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1. Introduction

Multi-valued vector functions, or correspondences, play an important role in many areas of economics, including consumer and producer choice, general equilibrium, game theory, and mechanism design. For instance, proving existence of equilibrium typically rests on Kakutani's (1941) generalization of Brouwer's fixed-point theorem to correspondences. In a variety of optimization problems, *single*-valued solutions satisfy an integrability condition: they are gradients of convex (or concave) functions known as potentials.[1] Specifically, a single-valued function is the gradient of a differentiable potential if and only if it is conservative, i.e. its integral along every closed path vanishes. This result is known as the "Gradient Theorem" (also known as the fundamental theorem of calculus for line integrals, e.g. Stewart, 2003).

When is a correspondence equal to the subdifferential of a potential? Rockafellar (1970) shows that a necessary and sufficient condition is that the correspondence is maximal and cyclically monotone. To relate cyclical monotonicity to the more geometric notion of conservativeness, we build on Aumann's (1965) work on integrals of correspondences defined on the unit interval. The Aumann integral generally yields a convex set but we show that for a monotone correspondence this set is a singleton. We extend this uniqueness result to closed-path integrals of monotone correspondences in arbitrary dimensions, which allows us to define *conservativeness* – a monotone correspondence is conservative if and only if its integral along any closed path vanishes – and establish the *Subgradient Theorem* – a monotone correspondence is equal to the subdifferential of some potential if and only if it is conservative.

Conversely, integrating the conservative monotone correspondence enables us to recover the potential – the *Potential Theorem*. The usefulness of the Potential Theorem is that the dual of a convex (concave) potential is a concave (convex) potential and its subdifferential defines a maximal conservative monotone correspondence inverse to the original correspondence – the *Duality Theorem*. This theorem allows us to solve problems in one domain that seem intractable in the dual domain.

[1]For instance, Hotelling's (1932) lemma dictates that the supply function is the gradient of the profit function. Shephard's (1970) lemma dictates that Hicksian demand is the gradient of the expenditure function and that factor demand is the gradient of the cost function. Finally, with quasilinear preferences, Roy's (1947) identity dictates that Marshallian demand is minus the gradient of the indirect utility function, see Section 3. In mechanism design, the optimal allocation rule often corresponds to the gradient of a convex potential, see e.g. Rochet and Choné (1998); Manelli and Vincent (2007); Goeree and Kushnir (2023).

We use our duality result in an application that is based on an inspiring recent contribution by Baldwin and Klemperer (2019) who use insights from tropical geometry to study consumer choice when goods are indivisible and preferences are quasilinear. They show that the graph of Marshallian demand in price space forms a polyhedral complex, the "price complex," and that its facets, on which demand is multi-valued, form a tropical curve.[2] Conversely, they show that a price complex arises from maximization of a concave utility if and only if the tropical curve formed by its weighted facets is "balanced" (Mikhalkin, 2004). Finally, they show that the price complex is dual to a polyhedral complex in quantity space, the "demand complex," but note that its nature is more "abstract" in the sense that it is associated to a class of utility functions rather than a specific valuation.

We show that the geometric duality outlined by Baldwin and Klemperer (2019) is a reflection of the usual duality between demand and inverse demand and that balancedness of their price and demand complexes is a reflection of conservativeness of demand and inverse demand. Specifically, we show that the price complex is equivalent to the subdifferential of indirect utility – the *Equivalence Theorem* – and that Mikhalkin's (2004) balancing condition is a tropical version of conservativeness. Since the inverse of a subdifferential correspondence is itself a subdifferential correspondence, it is equivalent to another polyhedral complex dual to the original. This dual complex is equivalent to the subdifferential of utility and generalizes Baldwin and Klemperer's (2019) demand complex in that it is not "abstract," but contains the same information as the price complex. We relate our findings to classical results in consumer demand theory: in price space, demand follows from Roy's (1947) identity and in quantity space inverse demand follows from the usual premise that marginal utilities equal prices.

Section 2 defines conservative monotone correspondences and derives the Subgradient, Potential, and Duality Theorems. Section 3 reexamines demand with indivisible goods and derives the Equivalence Theorem. Section 4 concludes. The Appendix contains the proofs.

[2]A polyhedral complex is a collection of polyhedra that form a partition of the space. The tropical graph is where the polyhedra meet.

2. Conservative Monotone Correspondences

We assume agents have quasilinear preferences $U(\mathbf{q}) - \mathbf{p}\cdot\mathbf{q}$ where the utility $U(\mathbf{q})$ over bundles is concave. The indirect utility $V(\mathbf{p}) = \max_{\mathbf{q}} U(\mathbf{q}) - \mathbf{p}\cdot\mathbf{q}$ is convex and determines demand $\mathbf{Q}(\mathbf{p})$ via Roy's lemma. However, since indirect utility is not necessarily everywhere differentiable, e.g. when goods are indivisible as in the next section, we need a generalized notion of derivatives.

Let $C \subseteq \mathbb{R}^n$ be a convex set and let $V : C \to \mathbb{R}$ be a convex function. Recall that $\mathbf{v} \in \mathbb{R}^n$ is a *subgradient* of V at $\mathbf{p} \in C$ if for all $\mathbf{p}' \in C$

$$V(\mathbf{p}') - V(\mathbf{p}) \geq \mathbf{v}\cdot(\mathbf{p}' - \mathbf{p})$$

The set of all subgradients at $\mathbf{p}$ is called the *subdifferential* of V at $\mathbf{p}$ and is denoted $\partial V(\mathbf{p})$. This set is non-empty, convex, and compact for any $\mathbf{p} \in C$. Below we demonstrate that $\partial V : C \to \mathbb{R}^n$ is a maximal conservative monotone correspondence.

Recall that a correspondence $\mathbf{M} : C \to \mathbb{R}^n$ is *monotone* if and only if

$$(\mathbf{m}_2 - \mathbf{m}_1)\cdot(\mathbf{p}_2 - \mathbf{p}_1) \geq 0$$

for all $\mathbf{p}_1, \mathbf{p}_2 \in C$, $\mathbf{m}_1 \in \mathbf{M}(\mathbf{p}_1)$, $\mathbf{m}_2 \in \mathbf{M}(\mathbf{p}_2)$. A correspondence is maximal if its graph is not properly contained in the graph of another monotone correspondence.[3]

We next show that a monotone correspondence is conservative if and only if it is equal to the subdifferential of some convex function – called the *potential*. We characterize this potential and establish properties of its dual. These are novel results in the mathematics literature and in Section 3 we apply them to study important economic environments where differentiability cannot be assumed.

Definition 1 *A maximal monotone correspondence,* $\mathbf{M}$*, is conservative if and only if, for any closed path* $\Gamma \subset C$ *where* C *is a convex subset of* $\mathbb{R}^n$*,*

$$\oint_{\Gamma} \mathbf{M}(\mathbf{p})\cdot d\mathbf{p} = 0 \tag{1}$$

We show in Appendix A that the path integral in (1) is unique, i.e. independent of the selection from $\mathbf{M}$ used to compute it, and therefore (1) is well defined.

[3]E.g., a univariate $\mathbf{M}$ is maximal if its graph can be drawn without lifting one's pen from the paper.

Theorem 1 (Subgradient Theorem) *If $V : C \to \mathbb{R}$ is convex then $\mathbf{M} = \partial V$ is a maximal conservative monotone correspondence on C. Conversely, if $\mathbf{M}$ is a maximal conservative monotone correspondence on C then $\mathbf{M} = \partial V$ for some convex $V : C \to \mathbb{R}$.*

Remark 1 To the best of our knowledge Theorem 1 has not been stated in the literature. However, its proof (see Appendix) can be obtained by combining prior work. The "if" part follows from Krishna and Maenner (2001, Th. 1). The "converse" part requires several steps. First, Kenderov (1975) showed that the integral of a maximal monotone correspondence $M(t)$ on the unit interval $t \in [0,1]$ is single valued and independent of the selection used to calculate it. Second, this uniqueness result can be extended to the integral of a maximal monotone correspondence $\mathbf{M} : C \to \mathbb{R}^n$ along the line segment from $\mathbf{p}_1 \in C$ to $\mathbf{p}_2 \in C$ by parameterizing this segment as $\Gamma(\mathbf{p}_1, \mathbf{p}_2) = (1-t)\mathbf{p}_1 + t\mathbf{p}_2$ for $t \in [0,1]$. Uniqueness maintains when combining several line segments to form a closed polyline, i.e. $\Gamma = \Gamma(\mathbf{p}_1, \mathbf{p}_2) \cup \Gamma(\mathbf{p}_2, \mathbf{p}_3) \cup \ldots \cup \Gamma(\mathbf{p}_K, \mathbf{p}_1)$. Summing the results for the different segments and using monotonicity of $\mathbf{M}(\mathbf{p})$ yields the inequality

$$\sum_{k=1}^{K} \mathbf{m}_k \cdot (\mathbf{p}_{k+1} - \mathbf{p}_k) \geq \oint_\Gamma \mathbf{M}(\mathbf{p}) \cdot d\mathbf{p}$$

for $\mathbf{p}_{K+1} = \mathbf{p}_1$ and any $\mathbf{m}_k \in \mathbf{M}(\mathbf{p}_k)$. If $\mathbf{M}(\mathbf{p})$ is conservative the right side is zero and the inequality implies $\mathbf{M}(\mathbf{p})$ is cyclically monotone. Rockafellar (1970, Th. 24.8, 24.9) shows that $\mathbf{M}(\mathbf{p})$ is a maximal cyclically monotone correspondence if and only if $\mathbf{M}(\mathbf{p}) = \partial V(\mathbf{p})$ for some convex V. ■

Remark 2 Convex functions are differentiable almost everywhere so the gradient ∇V is the unique selection from ∂V almost everywhere, see Rockafellar (1970, Thm. 25.5) and Rockafellar and Wets (2009, Thm. 9.60). But one cannot use the gradient theorem to establish conservativeness because the path Γ might be partly, or completely, contained in a lower-dimensional set of non-differentiability. For instance, if $V(p_1, p_2) = \max(p_1^2 + p_2^2, 1)$ and Γ is the unit circle, then the gradient does not exist anywhere along the path. However, the difference between any two selections from $\partial V(p_1, p_2) = \alpha(p_1, p_2)$ for $0 \leq \alpha \leq 2$ is a vector normal to the path. So while ∂V is multi-valued, the inner product of ∂V with the path's tangent is *single valued* along the entire path. Hence, different selections yield the same result for the path integral in (1). See the proof of Theorem 1 for details. ■

We can be more specific about the "some convex V" in the converse part of Theorem 1.

Pick some $\mathbf{p}_1 \in C$ and some finite value for $V(\mathbf{p}_1)$. Let $\Gamma(\mathbf{p}_1,\mathbf{p}_2)$ denote a path from $\mathbf{p}_1$ to $\mathbf{p}_2$. If $\mathbf{M}$ is conservative then

$$V(\mathbf{p}_2) = V(\mathbf{p}_1) + \int_{\Gamma(\mathbf{p}_1,\mathbf{p}_2)} \mathbf{M}(\mathbf{p}) \cdot d\mathbf{p} \tag{2}$$

is independent of $\Gamma(\mathbf{p}_1,\mathbf{p}_2)$. This function is known as the *potential* for $\mathbf{M}$.

Theorem 2 (Potential Theorem) *If* $\mathbf{M}$ *is a maximal conservative monotone correspondence then* $\mathbf{M} = \partial V$ *with the potential* V *defined in* (2).

An important property of subdifferential mappings is that they can be inverted in the sense of multi-valued mappings. If $\mathbf{M} = \partial V$ then there exists another maximal monotone correspondence, $\mathbf{M}^* = \partial V^*$, such that $\mathbf{p} \in \mathbf{M}^*(\mathbf{m})$ if and only if $\mathbf{m} \in \mathbf{M}(\mathbf{p})$, see Rockafellar (1970, Cor. 23.5.1). Here $V^*(\mathbf{m}) = \max_{\mathbf{p}} \langle \mathbf{p} | \mathbf{m} \rangle - V(\mathbf{p})$ is the Fenchel dual of V. Since $\mathbf{M}^*$ is a subdifferential mapping it is a maximal conservative monotone correspondence by Theorem 1. We thus have:

Theorem 3 (Duality) *Let* $\mathbf{M}$ *be a maximal conservative monotone correspondence with potential* V. *Then its inverse* $\mathbf{M}^*$ *is a maximal conservative monotone correspondence with potential* V^*, *which is the Fenchel dual of* V.

Theorem 3 expresses duality between two convex potentials. In economics, the convention is to define duality between a *concave* utility and a *convex* indirect utility. In particular, for a concave utility $U(\mathbf{q})$ the indirect utility $V(\mathbf{p}) = \max_{\mathbf{q}} U(\mathbf{q}) - \mathbf{p} \cdot \mathbf{q}$ is convex. Conversely, for a convex indirect utility $V(\mathbf{p})$ the dual

$$U(\mathbf{q}) = \min_{\mathbf{p}} V(\mathbf{p}) + \langle \mathbf{p} \,|\, \mathbf{q} \rangle \tag{3}$$

is the original concave utility.[4] The associated conservative monotone correspondences are the Marshallian demand correspondence $\mathbf{Q}(\mathbf{p}) = -\partial V(\mathbf{p})$ and the inverse demand correspondence $\mathbf{P}(\mathbf{q}) = \partial U(\mathbf{q})$.[5] In the next section, we study these correspondences in economics problems with indivisibilities, e.g. the assignment of a discrete set of goods. In such problems, the potentials are typically finitely generated, or polyhedral.

[4] It is readily verified that the two duality notions are related via $U(\mathbf{q}) = -V^*(-\mathbf{q})$.

[5] Some authors use the terminology superdifferential for concave functions. We follow Rockafellar (1970) who uses subdifferential for both convex and concave functions.

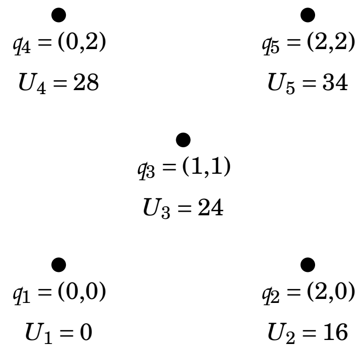

Figure 1: Example of a finite bundle set, $Q = \{q_k\}_{k=1}^{5}$, where the value of q_k is U_k.

3. Economies with Indivisible Goods

Economies with indivisible goods pose an obvious technical challenge, even with a single consumer. The usual approach of equating marginal rates of substitution to price ratios seems impossible with only a finite set of values for a discrete set of bundles. To deal with this challenge, Baldwin and Klemperer (2019) use insights from the mathematics literature on tropical geometry. In this section we show that the duality they exploit, between demand in terms of prices and (inverse) demand in terms of quantities, fits squarely in the domain of convex analysis using our results from Section 2.[6] In particular, the geometric duality they outline is a reflection of the usual duality between demand and inverse demand and the balancedness of their price and demand complexes is a reflection of the conservativeness of demand and inverse demand.

We begin with an example to illustrate the equivalence between Baldwin and Klemperer's (2019) price and quantity complexes and the subdifferential mappings that yield demand and inverse demand. Consider the set of bundles shown in Figure 1 for which a consumer with quasilinear preferences has the indicated utilities. The indirect utility follows by simply enumerating over the possible bundles

$$V(\mathbf{p}) = \max(0, 16 - 2p_1, 24 - p_1 - p_2, 28 - 2p_2, 34 - 2p_1 - 2p_2) \tag{4}$$

and the demand function follows from Roy's identity $\mathbf{Q}(\mathbf{p}) = -\partial V(\mathbf{p})$. Since indirect utility is piecewise linear, its subdifferential is piecewise constant.

[6] The terminology "tropical analysis" reflects the overlap between results in tropical geometry and convex analysis.

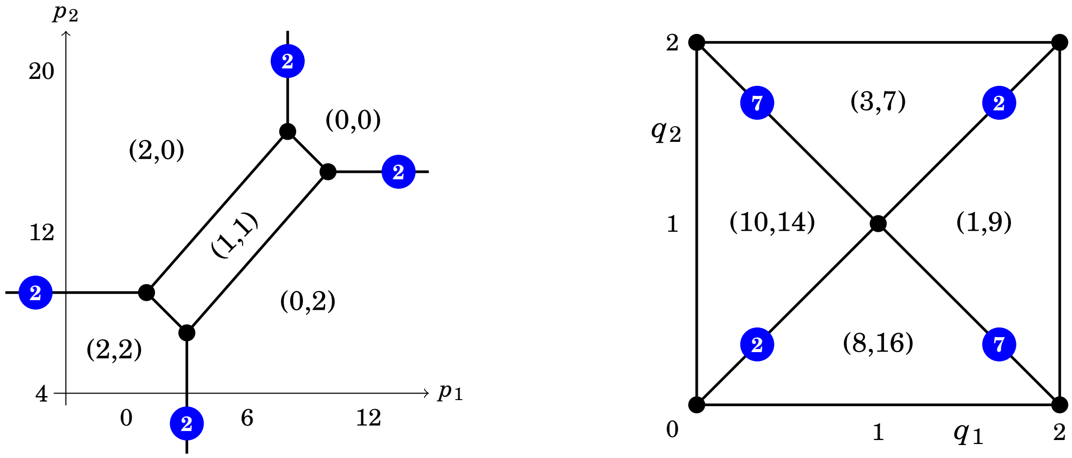

Figure 2: The left panel shows $-\partial V$ and the right panel shows ∂U. Vertices in one graph correspond to regions in the dual graph and their locations produce the labels of the dual regions. In both graphs, the difference in labels between two adjacent regions is the weight (the white number in the blue disk) times the normal to the regions' common edge. Edges in one graph correspond to perpendicular edges in the dual graph, and their weights equal the dual edge's length.

The left panel of Figure 2, coined the "price complex" by Baldwin and Klemperer (2019), shows the price regions where the demand correspondence is single-valued. The regions are separated by a "tropical graph," which contains prices at which demand is multi-valued.[7] Each region is labeled by the associated demand vector $-\partial V(\mathbf{p})$. The difference in labels between two adjacent regions is a vector perpendicular to the edge they share. The greatest common divisor of this difference vector defines the weight (indicated by the white number in the blue disk).[8] Dividing the difference vector by the weight yields a primitive integer normal vector whose entries have a greatest common divisor of 1. This primitive integer normal vector can be used to define lengths. For instance, the edge connecting the leftmost vertex $\mathbf{p}_1 = (1,9)$ with the topmost vertex $\mathbf{p}_2 = (8,16)$ in the left panel of Figure 2 has length seven since the difference $\mathbf{p}_2 - \mathbf{p}_1$ is seven times the integer normal vector $(1,1)$.

The dual $U(\mathbf{q})$ is the lowest concave function everywhere above the discrete valuations, i.e. $U(\mathbf{q}) \geq U_{\mathbf{q}}$ for all $\mathbf{q} \in Q$. It is given by

$$U(\mathbf{q}) = \min(14 + q_1 + 9q_2, 14 + 3q_1 + 7q_2, 8q_1 + 16q_2, 10q_1 + 14q_2) \quad (5)$$

[7] Baldwin and Klemperer (2019) refer to this tropical graph as the "locus of indifference prices." At these prices, demand equals $-\partial V(p) \cap \mathbb{Z}^K_{\geq 0}$ to reflect indivisibility.

[8] For edges without a blue disk the weight is equal to one.

for $\mathbf{q} = (q_1, q_2) \in [0,2]^2$. The resulting price function $\mathbf{P}(\mathbf{q}) = \partial U(\mathbf{q})$ yields the "demand complex" shown in the right panel of Figure 2. This graph shows bundles of demand among which the consumer is indifferent for some price vector. Each region of indifference is annotated with the price vector $\mathbf{P}(\mathbf{q}) = \partial U(\mathbf{q})$ that induces this indifference. As explained in the figure's caption, the graphs in the left and right panels are dual, which is a consequence of the fact that $-\partial V(\mathbf{p})$ and $\partial U(\mathbf{q})$ are inverses in the sense of multi-value mappings, i.e. $\mathbf{q} \in -\partial V(\mathbf{p})$ if and only if $\mathbf{p} \in \partial U(\mathbf{q})$.

The price complex in the left panel of Figure 2 coincides with the description of Baldwin and Klemperer (2019) but the demand complex in the right panel adds detail to their abstract version. Baldwin and Klemperer (2019, p. 881) write that the price complex "shows the actual prices at which bundles are demanded, whereas a demand complex shows only collections of bundles among which the agent is indifferent for some prices," and that there does not "seem to be any simple check of which polyhedral complexes in quantity space correspond to any valuation."

However, the inverse nature of the price and demand complexes implies they contain the same information. Hence, the demand complex *does* indicate the prices at which a consumer demands certain bundles. For example, at the price vector $(8,16)$, the consumer is different between bundles $(0,0)$, $(1,1)$ and $(2,0)$, see the lower triangle in the right panel of Figure 2; for any price vector that is a strict convex combination of $(8,16)$ and $(10,14)$, the consumer is indifferent between bundles $(0,0)$ and $(1,1)$; at any price vector that is a strict convex combination of $(8,16)$, $(10,14)$, $(3,7)$ and $(1,9)$, the consumer demands the bundle $(1,1)$. This shows that the prices at which bundles are demanded can be inferred from the demand complex.

Moreover, Baldwin and Klemperer's (2019, Th. 2.14) criterion for whether a price complex stems from utility maximization is that its facet subcomplex satisfies a balancedness condition due to Mikhalkin (2004).[9] To illustrate, consider a tiny circle around the leftmost vertex $(1,9)$ in left panel of Figure 2, which is at the intersection of the $(2,0)$, $(2,2)$, and $(1,1)$ regions. This circle crosses three edges. Balancedness

[9]The question whether demand can be rationalized by utility maximization has a long history in economics (dating back to Afriat, 1967) and has been raised across a variety of contexts. For instance, Green and Park (1996) analyze whether alternative models for choice under uncertainty yield contingent plans that can be rationalized by maximization of conditional expected utility.

requires that the edges' weighted normals sum to zero, i.e.

$$2\begin{pmatrix}0\\1\end{pmatrix}+\begin{pmatrix}1\\-1\end{pmatrix}+\begin{pmatrix}-1\\-1\end{pmatrix}=\mathbf{0}$$

However, the same balancedness condition applies to the demand complex. Starting at the top and going counterclockwise around the $(1,1)$ vertex in the right panel of Figure 2 we have

$$7\begin{pmatrix}1\\1\end{pmatrix}+2\begin{pmatrix}-1\\1\end{pmatrix}+7\begin{pmatrix}-1\\-1\end{pmatrix}+2\begin{pmatrix}1\\-1\end{pmatrix}=\mathbf{0}$$

The reason the demand complex *is* balanced is that $\mathbf{P}(\mathbf{q})$ is conservative by Theorem 1, i.e. $\oint_\Gamma \mathbf{P}(\mathbf{q})\cdot d\mathbf{q}=0$ along any closed path Γ. To summarize, indivisible goods pose no problem for solving the consumer's problem, either in price space via Roy's lemma or in quantity space by equating marginal utilities to prices. The reason we can use either space is that the price and demand complexes contain the same information.

To show equivalence of the price (demand) complex and the subdifferential of indirect (direct) utility more generally we need two definitions. Recall that a polyhedron is the intersection of finitely many half-spaces.[10]

Definition 2 *A polyhedral complex* Π *is a finite set of polyhedra in* $\mathbb{R}^n$ *such that:*

(i) if $P\in\Pi$ *then any face of* P *is also in* Π*;*

(ii) if $P,Q\in\Pi$ *then* $P\cap Q$ *is a face of both* P *and* Q*.*

Π *is a polyhedral subdivision of* $\mathbb{R}^n$ *if the union of its polyhedra covers* $\mathbb{R}^n$*.*

Definition 3 *The pair* (Π,Λ) *defines a* normally labeled *polyhedral subdvision of* $\mathbb{R}^n$ *if* Π *is a polyhedral subdivision of* $\mathbb{R}^n$ *and* Λ *a set of labels, one label* $\boldsymbol{\ell}_P\in\mathbb{R}^n$ *for each* $P\in\Pi$*, such that for all* $P,Q\in\Pi$ *with* $F=P\cap Q\neq\emptyset$ *(i.e.* P *and* Q *are adjacent and share a facet* F*), the vector* $\boldsymbol{n}=\boldsymbol{\ell}_Q-\boldsymbol{\ell}_P$ *points from* Q *to* P *and is normal to* F*.*

Definition 3 allows for non-integer labels so that we can establish a more general equivalence result. Recall that $V:\mathbb{R}^n\to\mathbb{R}$ is a polyhedral convex function if it is the pointwise

[10] A face of a polyhedron $P\subsetneq\mathbb{R}^n$ is any intersection of P with the boundary of a closed half-space that contains P; the dimension of a face is the dimension of its affine hull; 0-dimensional faces are points in $\mathbb{R}^n$ called vertices – throughout we identify a vertex with its position in $\mathbb{R}^n$; 1-dimensional faces are called edges; the empty set is also a face of P with dimension -1; a proper face of P is a face that is not P itself; a facet is a maximal proper face; the boundary ∂P of P is the union of its facets; if P is n-dimensional then its interior is $\mathrm{int}(P)=P\setminus\partial P$, otherwise its relative interior is $\mathrm{relint}(P)=P\setminus\partial P$.

maximum of a finite set of affine functions. We say $V(\mathbf{p})$ is irreducible if each of the affine functions exceeds the others for some $\mathbf{p} \in \mathbb{R}^n$.

Theorem 4 (Equivalence) *A normally labeled polyhedral subdivision of $\mathbb{R}^n$ is equivalent to the subdifferential of an irreducible polyhedral convex function in that:*

(i) An irreducible polyhedral convex function $V(\mathbf{p}) = \max_{\boldsymbol{\ell} \in \Lambda}(c_{\boldsymbol{\ell}} - \boldsymbol{\ell} \cdot \mathbf{p})$, with $\Lambda \subset \mathbb{R}^n$ finite, generates the normally labeled polyhedral subdvision $(\Pi, \Lambda) = \{(P_{\boldsymbol{\ell}}, \boldsymbol{\ell})\}_{\boldsymbol{\ell} \in \Lambda}$ where the

$$P_{\boldsymbol{\ell}} = \{\mathbf{p} \in \mathbb{R}^n \,|\, \boldsymbol{\ell} \in -\partial V(\mathbf{p})\} \tag{6}$$

are n-dimensional polyhedra that subdivide $\mathbb{R}^n$.

(ii) Conversely, let (Π, Λ) be a normally labeled polyhedral subdivision of $\mathbb{R}^n$ then it is generated by the irreducible polyhedral convex function

$$V(\mathbf{p}) = \max_{P \in \Pi}(c_P - \boldsymbol{\ell}_P \cdot \mathbf{p}) \tag{7}$$

where the constants $\{c_P\}_{P \in \Pi}$ satisfy $c_P - c_Q = \boldsymbol{\ell}^* \cdot (\boldsymbol{\ell}_P - \boldsymbol{\ell}_Q)$ whenever $P, Q \in \Pi$ are adjacent and $\boldsymbol{\ell}^*$ is any vertex in $P \cap Q$.

Theorem 4 only specifies the differences $c_P - c_Q$ so the convex potential in (7) is determined up to a constant. We can similarly determine, up to a constant, the concave potential for the dual demand complex (Π^*, Λ^*). Let $\Lambda^*(P)$ denote the set of vertices of $P \in \Pi$ and $\Lambda^* = \cup_{P \in \Pi} \Lambda^*(P)$. Define $U : \text{co}(\boldsymbol{\ell}) \to \mathbb{R}$ as follows[11]

$$U(\mathbf{q}) = \min_{\boldsymbol{\ell}^* \in \Lambda^*}(c_{\boldsymbol{\ell}^*} + \boldsymbol{\ell}^* \cdot \mathbf{q}) \tag{8}$$

where the constants $\{c_{\boldsymbol{\ell}^*}\}_{\boldsymbol{\ell}^* \in \Lambda^*}$ satisfy $c_{\boldsymbol{\ell}^*} - c_{\boldsymbol{r}^*} = \boldsymbol{\ell} \cdot (\boldsymbol{r}^* - \boldsymbol{\ell}^*)$ if there is a $P \in \Pi$ containing both $\boldsymbol{\ell}^*$ and $\boldsymbol{r}^*$, and $\boldsymbol{\ell}$ is the label of any such P. Then (Π^*, Λ^*) is generated by $\partial U(\mathbf{q})$.

4. Conclusions

We establish the Subgradient Theorem, which is an extension of the well known Gradient Theorem to monotone correspondences. Specifically, we show that any maximal

[11] The dual potential U is assumed to be $-\infty$ outside of the convex hull of the labels, $\text{co}(\boldsymbol{\ell})$.

conservative monotone correspondence has a convex or concave potential and, conversely, that the subdifferential of any convex or concave potential defines a maximal conservative monotone correspondence. Further, our Potential Theorem shows how to construct the potential from the correspondence, and our Duality Theorem shows that the inverse of the correspondence is also a maximal conservative monotone correspondence with a potential that is the dual of the potential of the original correspondence.

Our results allow us to apply the tools of convex analysis to economies with indivisible goods, generating analogues to classic results including Roy's identity and equating marginal utilities to prices. Moreover, it enables a reinterpretation of the important results of Baldwin and Klemperer (2019), couching their duality results and insights within the familiar realm of convex analysis. This allowed us to sharpen their notion of a demand complex to include the prices at which certain bundles are demanded.

A. Appendix: Proofs

We adapt Aumann's (1965) definition for integrals of multi-valued maps defined on the unit interval. For each $t \in [0,1]$, let $M(t)$ be a nonempty bounded subset of $\mathbb{R}$. We say that a single-valued function $m : [0,1] \to \mathbb{R}$ is a *measurable selection* from M if it is integrable and $m(t) \in M(t)$ for all $t \in [0,1]$. The Aumann integral of M is then defined as

$$\int_0^1 M(t)dt = \left\{ \int_0^1 m(t)dt : m \text{ is a measurable selection from } M \right\}$$

Generally, the right side yields a non-empty convex set (Aumann, 1965). Our main interest, however, is in *monotone* correspondences, i.e. $(m_2 - m_1)(t_2 - t_1) \geq 0$ for all $m_1 \in M(t_1)$, $m_2 \in M(t_2)$, $t_1, t_2 \in [0,1]$. Throughout we assume that the correspondence is maximal, i.e. its graph is not properly contained in the graph of another monotone correspondence, and comment on how results change if it is not. Kenderov (1975, Th. 2.7) shows that the set on which a maximal monotone correspondence is multi-valued has measure zero. Different selections from M are therefore equal almost everywhere and are continuous almost everywhere (as their set of discontinuity points coincides with the set where M is multi-valued). Since $M(t)$ is non-empty and bounded for all $t \in [0,1]$, each selection is also bounded, and hence, it is Riemann integrable. To summarize, the Aumann integral of a monotone correspondence on [0,1] is unique, i.e. a singleton, and we can use any selection from M to compute it. Finally, since $M(t)$ is monotone we have

$$m_0 \leq \int_0^1 M(t)dt \leq m_1$$

for all $m_0 \in M(0)$ and $m_1 \in M(1)$.

Following Romano et al. (1993), we next extend uniqueness of the Aumann integral to line integrals of maximal monotone correspondences $\mathbf{M}: C \to \mathbb{R}^n$ defined on some convex domain $C \subseteq \mathbb{R}^n$.[12] Recall that $\mathbf{M}$ is *monotone* if and only if

$$(\mathbf{m}_2 - \mathbf{m}_1) \cdot (\mathbf{p}_2 - \mathbf{p}_1) \geq 0$$

for all $\mathbf{p}_1, \mathbf{p}_2 \in C$, $\mathbf{m}_1 \in \mathbf{M}(\mathbf{p}_1)$, $\mathbf{m}_2 \in \mathbf{M}(\mathbf{p}_2)$. For $\mathbf{p}_1, \mathbf{p}_2 \in C$, the projection of $\mathbf{M}$ along the line segment $\mathbf{p}(t) = (1-t)\mathbf{p}_1 + t\mathbf{p}_2$ defines a correspondence on $[0,1]$

$$M(t) = \{\mathbf{m}(t) \cdot (\mathbf{p}_2 - \mathbf{p}_1) : \mathbf{m}(t) \in \mathbf{M}(\mathbf{p}(t))\}$$

that is monotone. To see this, note that for all $m_1 \in M(t_1)$, $m_2 \in M(t_2)$ we have

$$(m_2 - m_1)(t_2 - t_1) = (\mathbf{m}(t_2) - \mathbf{m}(t_1)) \cdot (\mathbf{p}_2 - \mathbf{p}_1)(t_2 - t_1) = (\mathbf{m}(t_2) - \mathbf{m}(t_1)) \cdot (\mathbf{p}(t_2) - \mathbf{p}(t_1)) \geq 0$$

where the inequality follows from monotonicity of $\mathbf{M}$. The inequality $(m_2 - m_1)(t_2 - t_1) \geq 0$ for all $m_1 \in M(t_1)$, $m_2 \in M(t_2)$, $t_1, t_2 \in [0,1]$, is the same as the one of the previous paragraph where it was used to establish uniqueness. Hence, the line integral of $\mathbf{M}(\mathbf{p})$ from $\mathbf{p}_1$ to $\mathbf{p}_2$ is uniquely defined, i.e. independent on the choice of $\mathbf{m}(t) \in \mathbf{M}(\mathbf{p}(t))$. Moreover,

$$\mathbf{m}_1 \cdot (\mathbf{p}_2 - \mathbf{p}_1) \leq \int_{\mathbf{p}_1}^{\mathbf{p}_2} \mathbf{M}(\mathbf{p}) \cdot d\mathbf{p} \leq \mathbf{m}_2 \cdot (\mathbf{p}_2 - \mathbf{p}_1)$$

for all $\mathbf{m}_1 \in \mathbf{M}(\mathbf{p}_1)$ and $\mathbf{m}_2 \in \mathbf{M}(\mathbf{p}_2)$.

Finally, the integral of $\mathbf{M}$ along a closed path, Γ, made out of a finite number of line segments is the sum of the integrals for each of the segments. For $k = 1, \ldots, K$ (with K arbitrary), let $\mathbf{p}_k$ denote the start of segment k and let $\mathbf{p}_{k+1}$ denote its end, with $\mathbf{p}_{K+1} \equiv \mathbf{p}_1$ and $\mathbf{m}_{K+1} \equiv \mathbf{m}_1$. We have

$$\sum_{k=1}^{K} \mathbf{m}_k \cdot (\mathbf{p}_{k+1} - \mathbf{p}_k) \leq \oint_{\Gamma} \mathbf{M}(\mathbf{p}) \cdot d\mathbf{p} \leq \sum_{k=1}^{K} \mathbf{m}_{k+1} \cdot (\mathbf{p}_{k+1} - \mathbf{p}_k)$$

If the closed-path integral of $\mathbf{M}$ along the polyline Γ vanishes then the left inequality is the definition of $\mathbf{M}$ being *cyclically monotone*.[13] Rockafellar (1970, Th. 24.8, 24.9) shows that $\mathbf{M}$ is cyclically monotone if and only if $\mathbf{M}(\mathbf{p}) \subseteq \partial F(\mathbf{p})$ for some convex function F with equality when $\mathbf{M}$ is maximal. Krishna and Maenner (2001) prove that the integral of the subdifferential of a convex function along any closed path[14] vanishes. Combining these results allows us to define conservative monotone correspondences and state the Subgradient Theorem.[15]

[12]If $\mathbf{M}$ is not maximal then the existence of a maximal extension of is ensured by Zorn's lemma.

[13]And the right inequality, which can be rewritten as $\sum_{k=1}^{K} \mathbf{x}_k \cdot (\mathbf{m}_{k+1} - \mathbf{m}_k) \leq 0$, implies that the inverse of $\mathbf{M}$ is also cyclically monotone.

[14]When we write path we will implicitly assume it is differentiable almost everywhere.

[15]Note that if the integral of $\mathbf{M}$ along any closed polyline vanishes then its integral along any closed path vanishes. This might also be shown by approximating an arbitrary closed path by closed polylines with increasingly many segments.

Proof of Theorem 1. Rockafellar (1970, Th. 24.9 and Cor. 31.5.2) shows that ∂V is maximal cyclically monotone and maximal monotone. Krishna and Maenner (2001) provide an elegant proof that ∂V is conservative. Here we provide a slightly different proof based on the intuition that for any $\mathbf{m}_1, \mathbf{m}_2 \in \partial V(\mathbf{p})$ their difference is normal to the curve that passes through $\mathbf{p}$ almost everywhere. Recall that the directional derivative of V is defined as

$$V'(\mathbf{p};\mathbf{y}) = \lim_{\varepsilon \downarrow 0} \frac{V(\mathbf{p}+\varepsilon\mathbf{y}) - V(\mathbf{p})}{\varepsilon}$$

see Rockafellar (1970, Sec. 23) who shows that $V'(\mathbf{p};\mathbf{y})$ exists and $-V'(\mathbf{p};-\mathbf{y}) \le V'(\mathbf{p};\mathbf{y})$ for all $\mathbf{y}$, see Rockafellar (1970, Th. 23.1). Moreover, Rockafellar (1970, Th. 23.2) implies

$$-V'(\mathbf{p};-\mathbf{y}) \le \mathbf{m}\cdot\mathbf{y} \le V'(\mathbf{p};-\mathbf{y})$$

for all $\mathbf{y}$ iff $\mathbf{m}$ in $\partial V(\mathbf{p})$. Hence, for $\mathbf{m}_1, \mathbf{m}_2 \in -\partial V(\mathbf{p})$ we have

$$-V'(\mathbf{p};\mathbf{y}) - V'(\mathbf{p};-\mathbf{y}) \le (\mathbf{m}_2 - \mathbf{m}_1)\cdot\mathbf{y} \le V'(\mathbf{p};\mathbf{y}) + V'(\mathbf{p};-\mathbf{y})$$

Consider a path $\mathbf{p} : [0,1] \to C$ with $\mathbf{p}(0) = \mathbf{p}(1)$ that is differentiable almost everywhere. Define $\phi(t) = V(\mathbf{p}(t))$, which is regular Lipschitzian, so for almost all $t \in [0,1]$, $\phi(t)$ is differentiable, i.e. $\phi'(t;1) = -\phi'(t;-1)$, or, equivalently, $V'(\mathbf{p}(t);\dot{\mathbf{p}}(t)) = -V'(\mathbf{p}(t);-\dot{\mathbf{p}}(t))$. Hence, for almost all $t \in [0,1]$, the above inequality implies $(\mathbf{m}_2 - \mathbf{m}_1)\cdot\dot{\mathbf{p}}(t) = 0$ for any $\mathbf{m}_1, \mathbf{m}_2 \in \partial V(\mathbf{p}(t))$, i.e. the difference between two selections from the subdifferential is normal to the path almost everywhere. Hence, $\oint \partial V(\mathbf{p})\cdot d\mathbf{p} = \int_0^1 \partial V(\mathbf{p}(t))\cdot\dot{\mathbf{p}}(t)dt$ is independent of the selection from $\partial V(\mathbf{p})$ and since $\partial V(\mathbf{p}(t))\cdot\dot{\mathbf{p}}(t) = V'(\mathbf{p}(t);\dot{\mathbf{p}}(t)) = \phi'(t)$ for almost all $t \in [0,1]$, we have $\oint \partial V(\mathbf{p})\cdot d\mathbf{p} = \int_0^1 \phi'(t)dt = \phi(1) - \phi(0) = 0$, as first shown by Krishna and Maenner (2001).

For the proof of the converse part, consider the integral of $\mathbf{M}$ along a closed path, Γ, made out of a finite number of line segments. For $k = 1,\ldots,K$ (with K arbitrary), let $\mathbf{p}_k$ denote the start of segment k and let $\mathbf{p}_{k+1}$ denote its end, with $\mathbf{p}_{K+1} \equiv \mathbf{p}_1$. We have shown in the above

$$\sum_{k=1}^{K} \mathbf{m}_k\cdot(\mathbf{p}_{k+1} - \mathbf{p}_k) \le \oint_\Gamma \mathbf{M}(\mathbf{p})\cdot d\mathbf{p} \le \sum_{k=1}^{K} \mathbf{m}_{k+1}\cdot(\mathbf{p}_{k+1} - \mathbf{p}_k)$$

where $\mathbf{m}_k$ is any selection from $\mathbf{M}(\mathbf{p}_k)$ for $k = 1,\ldots,K$. If $\mathbf{M}$ is conservative, the path integral vanishes and we have

$$\sum_{k=1}^{K} \mathbf{m}_k\cdot(\mathbf{p}_{k+1} - \mathbf{p}_k) \le 0$$

for all $\mathbf{p}_k \in C$ and $\mathbf{m}_k \in \mathbf{M}(\mathbf{p}_k)$, which are the inequalities that define cyclical monotonicity of $\mathbf{M}$, see Rockafellar (1970, p. 238). Indeed, the idea behind choosing a closed polyline is to generate the above inequalities. Hence, $\mathbf{M}(\mathbf{p}) \subseteq \partial V(\mathbf{p})$ for some convex function V, and, if $\mathbf{M}$ is maximal then $\mathbf{M} = \partial V$, see (Rockafellar, 1970, Th. 24.8 and Th.

24.9). (Note that a vanishing closed polyline integral of $\mathbf{M}$ also implies

$$0 \le \sum_{k=1}^{K} \mathbf{m}_{k+1} \cdot (\mathbf{p}_{k+1} - \mathbf{p}_k) = \sum_{k=1}^{K} \mathbf{p}_k \cdot (\mathbf{m}_{k+1} - \mathbf{m}_k)$$

which implies that the inverse $\mathbf{P}$ of $\mathbf{M}$ is also cyclically monotone and, hence, $\mathbf{P} \subseteq \partial U$ for some concave potential U with equality when $\mathbf{P}$ is maximal.) ■

An easy corollary is that a monotone correspondence is cyclically monotone if and only if it is conservative. (Note that every cyclically monotone correspondence is monotone, e.g. use $K = 2$ in the definition of cyclical monotonicity.)

Proof of Theorem 2. Let $\mathbf{p}_1 \in C$ and let $\Gamma(\mathbf{p}_1, \mathbf{p}_2)$ denote the line from $\mathbf{p}_1$ to $\mathbf{p}_2$. From definition of the potential (2) and the proof of Theorem 1 we have

$$V(\mathbf{p}_2) = V(\mathbf{p}_1) + \int_{\Gamma(\mathbf{p}_1,\mathbf{p}_2)} \mathbf{M}(\mathbf{p}) \cdot d\mathbf{p} \ge V(\mathbf{p}_1) + \mathbf{m}_1 \cdot (\mathbf{p}_2 - \mathbf{p}_1)$$

for all $\mathbf{p}_2 \in C$, $\mathbf{m}_1 \in \mathbf{M}(\mathbf{p}_1)$. Hence, $\mathbf{m_1} \in \partial V(\mathbf{p}_1)$ for all $\mathbf{p}_1 \in C$ and $\mathbf{m}_1 \in \mathbf{M}(\mathbf{p}_1)$, i.e. $\mathbf{M} \subseteq \partial V$ (with equality when $\mathbf{M}$ is maximal, see above). ■

Proof of Theorem 4. (i) $P_{\boldsymbol{\ell}}$ in (6) can equivalently be described as

$$\begin{aligned} P_{\boldsymbol{\ell}} &= \{\mathbf{p} \in \mathbb{R}^n \,|\, c_{\boldsymbol{\ell}} - \boldsymbol{\ell} \cdot \mathbf{p} \ge c_{\boldsymbol{\ell}'} - \boldsymbol{\ell}' \cdot \mathbf{p} \;\; \forall \boldsymbol{\ell}' \in \Lambda\} \\ &= \{\mathbf{p} \in \mathbb{R}^n \,|\, (\boldsymbol{\ell} - \boldsymbol{\ell}') \cdot \mathbf{p} \le c_{\boldsymbol{\ell}} - c_{\boldsymbol{\ell}'} \;\; \forall \boldsymbol{\ell}' \in \Lambda\} \end{aligned}$$

which is the standard definition of a polyhedron. It is n-dimensional because irreducibility of $V(\mathbf{p}) = \max_{\boldsymbol{\ell} \in \Lambda}(c_{\boldsymbol{\ell}} - \boldsymbol{\ell} \cdot \mathbf{p})$ means $c_{\boldsymbol{\ell}} - \boldsymbol{\ell} \cdot \mathbf{p} > c_{\boldsymbol{\ell}'} - \boldsymbol{\ell}' \cdot \mathbf{p} \;\forall \boldsymbol{\ell}' \neq \boldsymbol{\ell}$ for some $\mathbf{p} \in \mathbb{R}^n$ and these strict inequalities hold for a small n-dimensional open ball around $\mathbf{p}$.

Lower dimensional faces can be defined similarly, e.g. for $\boldsymbol{\ell}, \boldsymbol{\ell}' \in \Lambda$ the facets are $\mathscr{F}_{\boldsymbol{\ell},\boldsymbol{\ell}'} = \{\mathbf{p} \in \mathbb{R}^n \,|\, \boldsymbol{\ell}, \boldsymbol{\ell}' \in -\partial V(\mathbf{p})\}$ if non-empty. More generally, for $d = 1, \ldots, \min(n+1, |\Lambda|)$, let $\mathscr{S}_d$ be a set of d different labels. If $\mathscr{F}_{\mathscr{S}_d} = \{\mathbf{p} \in \mathbb{R}^n \,|\, \mathscr{S}_d \in -\partial V(\mathbf{p})\}$ is non-empty then it is a $(n+1-d)$-dimensional face in Π. For $k \le n$, the *k-skeleton* Π_k of Π is the set of all faces in Π of dimensional k or less, i.e. $\Pi_k = \cup_{d \ge n+1-k} \cup_{\mathscr{S}_d} \mathscr{F}_{\mathscr{S}_d}$. This defines a complex $\emptyset = \Pi_{-1} \subseteq \Pi_0 \subseteq \cdots \subseteq \Pi_n = \Pi$ that satisfies properties (i) and (ii) of Definition 2.[16] To verify the final property of Definition 2, note that for any $\mathbf{p} \in \mathbb{R}^n$, at least one term in $V(\mathbf{p})$ is the maximum so the $\{P_{\boldsymbol{\ell}}\}_{\boldsymbol{\ell} \in \Lambda}$ cover $\mathbb{R}^n$. Finally, to verify Definition 3, note that any $\mathbf{p}$ in a facet shared by two adjacent polyhedra $P_{\boldsymbol{\ell}}, P_{\boldsymbol{\ell}'} \in \Pi$ satisfies $c_{\boldsymbol{\ell}} - \boldsymbol{\ell} \cdot \mathbf{p} = c_{\boldsymbol{\ell}'} - \boldsymbol{\ell}' \cdot \mathbf{p}$. Hence, $(\boldsymbol{\ell} - \boldsymbol{\ell}') \cdot \mathbf{p} = c_{\boldsymbol{\ell}} - c_{\boldsymbol{\ell}'}$ for any $\mathbf{p}$ in the facet, which shows that $\boldsymbol{\ell} - \boldsymbol{\ell}'$ is normal to the facet and Π is normally labeled.

(ii) The constructed V is a polyhedral convex function. To show that $\boldsymbol{\ell}_P \in -\partial V(\mathbf{p})$ for $\mathbf{p} \in P$, it is sufficient that $c_P - \boldsymbol{\ell}_P \cdot \mathbf{p} \ge c_Q - \boldsymbol{\ell}_Q \cdot \mathbf{p}$ for all $Q \in \Pi$ that are adjacent to P (since V is convex).[17] The constants satisfy $c_P - c_Q = \boldsymbol{\ell}^* \cdot (\boldsymbol{\ell}_P - \boldsymbol{\ell}_Q)$ so we can rewrite

[16]Note that many of the Π_k may be empty, but not Π_n. For instance, if $V(\mathbf{p}) = \alpha - \boldsymbol{\beta} \cdot \mathbf{p}$ then $\Pi_n = \mathbb{R}^n$ and $\Pi_k = \emptyset$ for $k < n$.

[17]To elucidate, consider minimizing $\mu - \boldsymbol{\ell} \cdot p$ over the epigraph $\text{epi}(V) = \{(\mathbf{p}, \mu) \in \mathbb{R}^{n+1} \,|\, \mu \ge V(\mathbf{p})\}$, which

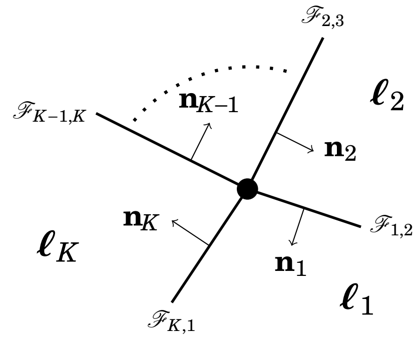

Figure 3: Balancedness and normal labeling.

this inequality as

$$(\mathbf{p}-\boldsymbol{\ell}^*)\cdot(\boldsymbol{\ell}_P-\boldsymbol{\ell}_Q)\leq 0 \tag{9}$$

Let ℓ_P^k and ℓ_Q^k denote the k-th entries of $\boldsymbol{\ell}_P$ and $\boldsymbol{\ell}_Q$ respectively. Recall from Definition 3 that the normal $\boldsymbol{n}=\boldsymbol{\ell}_Q-\boldsymbol{\ell}_P$ to the facet $P\cap Q$ points from Q to P. So, if $\ell_P^k\leq\ell_Q^k$ then $p_k\geq q_k$ and if $\ell_P^k\geq\ell_Q^k$ then $p_k\leq q_k$ for any $\mathbf{p}\in P$ and $\mathbf{q}\in Q$. Hence, $(\mathbf{p}-\mathbf{q})\cdot(\boldsymbol{\ell}_P-\boldsymbol{\ell}_Q)\leq 0$ for any $\mathbf{p}\in P$ and $\mathbf{q}\in Q$. Since $\boldsymbol{\ell}^*\in Q$ the condition in (9) is met.

To show the constructed V is irreducible, we sharpen (9) to $(\mathbf{p}-\boldsymbol{\ell}^*)\cdot(\boldsymbol{\ell}_P-\boldsymbol{\ell}_Q)<0$, or equivalently, $c_P-\boldsymbol{\ell}_P\cdot\mathbf{p}>c_Q-\boldsymbol{\ell}_Q\cdot\mathbf{p}$ for $\mathbf{p}\in\text{int}(P)$. Write $\mathbf{p}\in\text{int}(P)$ as $\mathbf{p}=\boldsymbol{\ell}^{**}+\alpha\mathbf{n}$ where $\boldsymbol{\ell}^{**}$ is a vector in the affine hull of $P\cap Q$ and $\alpha>0$. Then $(\mathbf{p}-\boldsymbol{\ell}^*)\cdot(\boldsymbol{\ell}_P-\boldsymbol{\ell}_Q)=-\alpha\,\mathbf{n}\cdot\mathbf{n}<0$.

Finally, we show that normal labeling implies Mikhalkin's (2004) balancing condition. Consider the configuration in Figure 3 where K facets with normals $\mathbf{n}_k$ intersect and the K adjacent regions are labeled $\boldsymbol{\ell}_k$ for $k=1,\ldots,K$. Normal labeling requires e.g. $\mathbf{n}_1=\boldsymbol{\ell}_2-\boldsymbol{\ell}_1$ or $\boldsymbol{\ell}_2=\boldsymbol{\ell}_1+\mathbf{n}_1$, and $\boldsymbol{\ell}_3=\boldsymbol{\ell}_2+\mathbf{n}_2$ etc. If we make a full counterclockwise circle we get

$$\boldsymbol{\ell}_1=\boldsymbol{\ell}_K+\mathbf{n}_K=\boldsymbol{\ell}_{K-1}+\mathbf{n}_K+\mathbf{n}_{K-1}=\cdots=\boldsymbol{\ell}_1+\sum_{k=1}^{K}\mathbf{n}_k$$

i.e. normal labeling implies the balancing condition[18] $\sum_{k=1}^K\mathbf{n}_k=\mathbf{0}$.

is a polyhedron in $\mathbb{R}^{n+1}$. This is a standard linear programming problem

$$\min_{\mu\geq c_P-\boldsymbol{\ell}_P\cdot\mathbf{p}\ \ \forall P\in\Pi}\ \mu-\boldsymbol{\ell}\cdot\mathbf{p}$$

and the simplex method used to solve it rests on the idea that a vertex in epi(V) is optimal when neighbor vertices do not yield lower values, which works if the feasible set is convex (like epi(V)). The intuition is that the vertex is a local minimum and convexity ensures it is a global minimum. For special values of the objective, e.g. when $\boldsymbol{\ell}=\boldsymbol{\ell}_P$, the solution to the minimization problem is not just a vertex but the entire polyhedron P_k. Nonetheless, optimality of any $\mathbf{p}\in P_k$ can be established by comparing its objective value to the best objective value obtained from adjacent polyhedra.

[18]The usual formulation of the balancing condition is that the sum of weighted normals vanishes, i.e. $\sum_{k=1}^K w_k\mathbf{n}_k=\mathbf{0}$, see Section 3. This is because labels are assumed to be integer and the difference $\boldsymbol{\ell}'-\boldsymbol{\ell}$ is divided by the greatest common divisor of its elements to define a primitive integer normal vector. However, there is no need to separate out weights in the generalization we consider, i.e. when labels are real numbers. Hence, the condition $\sum_{k=1}^K w_k\mathbf{n}_k=\mathbf{0}$ reduces to $\sum_{k=1}^K\mathbf{n}_k=\mathbf{0}$.

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